Juggling Theory Part I. Siteswap State Diagrams

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Abstract

The siteswap notation is a widely spread tool to describe juggling patterns [2, 3]. To find new siteswaps or transitions between two different siteswaps the so-called siteswap state diagrams were introduced [1, 2]. This paper deals with a new approach to compute siteswap state diagrams.

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1 Juggling States

Definition: Juggling States. We define the set of juggling states with b balls and a maximum throw height of $h \geq b$

$$\mathcal{S}(b,h) := \left\{ s \mid q_2(s) = b \wedge s < 2^h \right\} , \qquad (1)$$

where $q_2(s)$ is the digit sum of s in its binary representation.

The digit sum $q_m(s)$ of any number s in a base-m positional notation can be defined as

$$q_m(s) := \sum_{k=0}^{\lfloor \log_m s \rfloor} d_m(k,s) \,, \qquad (2a)$$

$$d_m(k,s) := \left\lfloor \frac{s}{m^k} \right\rfloor \bmod m \,, \qquad (2b)$$

where $d_m(k,s)$ is the k-th digit (from right) in base-m positional notation of s.

In the following we will note a state $s \in \mathcal{S}$ also in its binary representation. For a better readability, the zeros will be replaced by a dash (-). Leading zeros will also be written, so we can always see what our maximum throw height is.

By definition a juggling state $s \in \mathcal{S}(b,h)$ is in a binary h-digit number (includeing the leading zeros) and contains exact b ones. Therefore the total number of states n_s in a state space $\mathcal{S}(b,h)$ can be computed by the formula

$$n_\mathrm{s} = \binom{h}{b} \,. \qquad (3)$$

Table 1 shows the cardinal numbers of the set $\mathcal{S}(b,h)$, i. e. the total number of juggling states dependent on number

Table 1: Total number of states in dependence of number of balls b and maximum throw height h.

max. throw height h									
	1	1							
	2	2	1						
	3	3	3	1					
	4	4	6	4	1				
	5	5	10	10	5	1			
	6	6	15	20	15	6	1		
	7	7	21	35	35	21	7	1	
	8	8	28	56	70	56	28	8	1
		1	2	3	4	5	6	7	8
		number of balls b							

of balls b and maximum throw height h.

Definition: Ground State. The ground state s_{g} is defined as the smallest entry in the set $\mathcal{S}(b,h)$

$$\exists! s_{\mathrm{g}} \in \mathcal{S}(b,h) : s_{\mathrm{g}} \leq s \quad \forall s \in \mathcal{S}(b,h)\,. \tag{4}$$

The b-ball ground state s_{g} can be computed by

$$s_{\mathrm{g}} = \sum_{k=0}^{b-1} 2^k\,. \tag{5}$$

For example the ground state of 3-ball patterns up to height 5 will be denoted as `--111` $\in \mathcal{S}(3,5)$, which is a decimal 7. The 3-ball states up to height 5 are

$$\begin{aligned}\mathcal{S}(3,5) = \{7,11,13,14,19,21,22,25,26,28\} = \\ \{\texttt{--111},\texttt{-1-11},\texttt{-11-1},\texttt{-111-},\texttt{1--11}, \\ \texttt{1-1-1},\texttt{1-11-},\texttt{11--1},\texttt{11-1-},\texttt{111--}\}\,.\end{aligned} \tag{6}$$

2 Transitions

In this section we will introduce mappings. We will not use the classical notation $f(x)$, instead of that we will write $(x)f$, i. e. the mapping f operates on the number x. Furthermore we will use the composition of mappings $(x)(f \circ g) = ((x)f)g$ or in a shorter form $(x)fg$.

2.1 Elementary Transitions

To change juggling states, we need transitions between them. In practice, an elementary transition is a throw of an object.

Definition: Elementary Transitions. The set of all elementary transitions up to height h is defined as

$$\mathcal{T}(h) = \mathcal{T}_0 \cup \mathcal{T}_1(h) = \{0\} \cup \{1,2,3,\dots h\}\,. \tag{7}$$

The subsets $\mathcal{T}_k(h)$ are the possible throws with k objects. If you have no object you can just do one thing: throw nothing. If you have an object, you can throw it to an arbitrary height up to the maximum height h. Note that the elements of $\mathcal{T}(h)$ are mappings and no numbers. The corresponding number of $t \in \mathcal{T}(h)$ will be denoted as $|t|$.

Each elementary transition is a mapping between two states. It can be computed by

$$t : \mathcal{S}(b,h) \to \mathcal{S}(b,h)\,, \tag{8a}$$

$$t : s \mapsto \begin{cases} \frac{1}{2}s & : d_2(0,s) = 0 \\ \frac{1}{2}\left(s - 1 + 2^{|t|}\right) & : d_2(0,s) = 1 \end{cases}\,. \tag{8b}$$

If the last digit $d_2(0,s) = 0$, the state s will be shifted to the right (division by 2). Otherwise the last digit will be reset and the $|t|$-th digit will be set (if possible). Then the result has also to be shifted to the right.

Not every transition can operate on an arbitrary state. A transition $t \in \mathcal{T}(h)$ can operate on a state $s \in \mathcal{S}(b,h)$

if it satisfies the conditions

$$t \in \mathcal{T}_k(h)\,,\ \text{where } k = d_2(0,s)\,, \tag{9a}$$

$$d_2(|t|,s) = 0\,, \tag{9b}$$

i.e. the last digit $d_2(0,s)$ decides whether an object can be thrown ($k=1$) or not ($k=0$). If the object will be thrown at height $|t|$, the $|t|$-th digit has to be zero.

On the ground state `--111` of the state space $\mathcal{S}(3,5)$ only the transitions 3, 4 and 5 can operate. With the definition above we can compute the new state after a transition.

For example we start at state `--111` and perform the transition 3 on it:

$$(\texttt{--111})\,3 = \frac{1}{2}\left(\overbrace{\texttt{--111}}^{=7} - 1 + 2^3\right) = 7 = \texttt{--111}\,. \tag{10}$$

As we see, the transition 3 maps the state `--111` into itself.

2.2 Composed Transitions

Definition: Composed Transition. A composition of several elementary transitions is called composed transition. Given are n elementary transitions and $n+1$ states with

$$t_i : s_{i-1} \mapsto s_i \quad \forall i = 1 \ldots n\,, \tag{11}$$

then the composition of them is defined as the mapping

$$t_1 \circ t_2 \circ \cdots \circ t_n \equiv \bigcirc_{i=1}^{n} t_i : s_0 \mapsto s_n\,. \tag{12}$$

Note that the composition of transitions is not commutative, i.e. $t_1 t_2 \neq t_2 t_1$.

Definition: Siteswap. If a transition (elementary or composed transition) is an identity map

$$t \equiv \mathrm{id}_{\mathcal{S}(b,h)} : s \mapsto s\,, \tag{13}$$

i.e. it maps a state s into itself, it is called a “Siteswap”.

Most common siteswaps operate on the ground state s_g like 3 (cascade), 441 or 531. If a siteswap is an identity map on the ground state s_g, it is also called a “ground state pattern”. E.g. 441, 414 and 144 are the same tricks, but only 441 is a ground state pattern.

You can also arrange ground state patterns in an arbitrary way, e.g. ...441335314413... is a valid siteswap.

Theorem Siteswaps containing composed transitions $t_1 t_2$ with $|t_2| = |t_1| - 1 \geq 0$ are not valid.

Proof Assume that t_1 operates on s. Then the $|t_1|$-th digit of s will be set and shifted to the right by t_1. The $(|t_1|-1)$-th digit of the new state $s' = (s)t_1$ is now set, but t_2 operates only on s' if the $(|t_1|-1)$-th digit is not set. q.e.d.

2.3 Transition Matrices and State Diagrams

By determining all possible transitions, we can obtain state transition matrices and full state diagrams. Figure 1 shows all possible transitions between the states $s \in \mathcal{S}(3,5)$ in a state transition matrix. To find a valid siteswap using this matrix, choose an initial state and select a transition in this row. Go up or down to the diagonal of the matrix. Now you have found the next state. Repeat this until you are back on your initial state.

Figure 2 shows the same information in a full siteswap state diagram. A valid siteswap describes a closed curve in this graph.

If two siteswaps operate on different states like the cascade (siteswap 3) and the shower (siteswap 51), they cannot be combined directly, i.e. ...3335151... is not a valid siteswap. In this case, you have to find a transition between them. To find a valid transition from cascade to shower, you can have a look into the siteswap state diagram to find a path which connects both tricks. For

	--111	-1-11	-11-1	-111-	1--11	1-1-1	1-11-	11--1	11-1-	111--
--111	3	4			5					
-1-11	2		4			5				
-11-1	1			4			5			
-111-	0									
1--11		2	3					5		
1-1-1		1		3					5	
1-11-		0								
11--1			1	2						5
11-1-			0							
111--				0						

Figure 1: State transition matrix for 3 balls up to height 5 with some tricks: cascade (siteswap 3, red) and shower (siteswap 51, blue).

example valid transitions are

$$\ldots 333 \rightarrow \mathbf{4} \rightarrow 5151 \ldots$$
$$\ldots 333 \rightarrow \mathbf{52} \rightarrow 5151 \ldots$$
$$\ldots 333 \rightarrow \mathbf{5350} \rightarrow 5151 \ldots$$
$$\ldots 333 \rightarrow \mathbf{55150} \rightarrow 5151 \ldots$$

3 Reduced State Diagrams

As we can see in figure 2, there are some states which have just one input or one output. Now we will try to reduce the full state diagram by eleminating those trivial states.

Definition: Trivial State. A state s is called a trivial state, iff

$$\exists! t_1 \in \mathcal{T} : (s')t_1 = s \quad \vee \quad \exists! t_2 \in \mathcal{T} : (s)t_2 = s'' \qquad (14)$$

where s', s'' are arbitrary valid states. Or in other words, a state is called trivial if just one transition maps into it or just one transition can operate on it (or both).

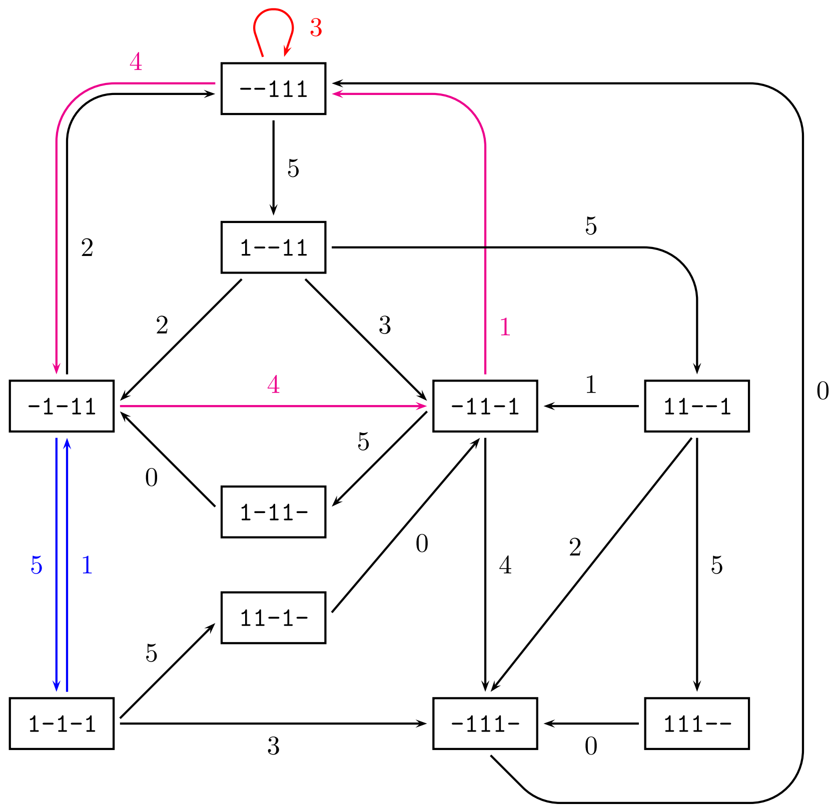

Figure 2: The full siteswap state diagram for 3 balls up to height 5 with some tricks: cascade (siteswap 3, red), shower (siteswap 51, blue) and 441 (magenta).

If a state s is a trivial state, we can omit it, when we draw the state transition matrix or the state diagram. But we cannot omit the composed transitions

$$(s')t_1t_2 = s'' , \quad \forall t_1, t_2 \in \mathcal{T} \text{ satisfying eq. (14)} . \qquad (15)$$

Now we have reduced the state s and therefore the composed transitions t_1t_2 can be examined as elementary transitions in further reduction.

Theorem All even states are trivial.

Proof A state s is even $\Leftrightarrow d_2(0, s) = 0$. By definition all transitions which can operate on an even state are in $\mathcal{T}_0 = \{0\}$. Therefore exists exact one transition that can operate on s. This fulfills the definition of trivial states. q.e.d.

Theorem All states $s > 2^{h-1}$ are trivial.

Proof By definition, the $(h-1)$-th digit of a state $s > 2^{h-1}$ is equal one. This is the highest digit of a state $s \in \mathcal{S}(b, h)$, so this digit cannot be generated by shifting a state $s' \in \mathcal{S}(b, h)$ to the right. The only way to generate this digit is to perform a throw t of maximum throw height $|t| = h$, which fulfills the definition of trivial states. q.e.d.

With the theorems above, a state $s \in \mathcal{S}(b, h)$ is trivial iff

$$d_2(0, s) = 0 \quad \vee \quad d_2(h, s) = 1 . \qquad (16)$$

Therefore a state $s \in \mathcal{S}(b, h)$ is nontrivial iff

$$d_2(0, s) = 1 \quad \wedge \quad d_2(h, s) = 0 . \qquad (17)$$

We know that a state $s \in \mathcal{S}(b, h)$ consists of h binary dig-

Table 2: Total number of nontrivial states s_{snt} in dependence of number of balls b and maximum throw height h.

max. throw height h	1	2	3	4	5	6	7
2	1						
3	1	1					
4	1	2	1				
5	1	3	3	1			
6	1	4	6	4	1		
7	1	5	10	10	5	1	
8	1	6	15	20	15	6	1

number of balls b

its. If s is nontrivial, two digits are already well-defined. To find nontrival states, we can only distribute $b-1$ ones among the inner $h-2$ digits. So the total number of nontrivial states n_{snt} can be computed by

$$n_{\text{snt}} = \binom{h-2}{b-1}. \tag{18}$$

Table 2 shows the number of nontrivial states in dependence of the number of balls b and the maximum throw height h.

Figure 3 shows the reduced state transition matrix and figure 4 shows the reduced siteswap state diagram. As we can see, there are only three nontrivial states in $\mathcal{S}(3,5)$. You can find much more reduced siteswap state diagrams for 3, 4 and 5 balls up to height 7 at [1].

	`--111`	`-1-11`	`-11-1`
`--111`	3, 5520, 55500	4, 52	53, 551
`-1-11`	2, 530	51	4, 550
`-11-1`	1, 40	50	

Figure 3: Reduced state transition matrix for 3 balls up to height 5 with some tricks: cascade (siteswap 3, red) and shower (siteswap 51, blue).

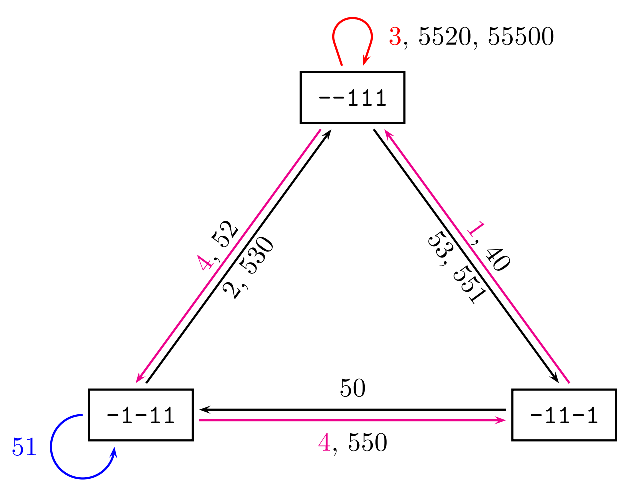

Figure 4: The reduced siteswap state diagram for 3 balls up to height 5 with some tricks: cascade (siteswap 3, red), shower (siteswap 51, blue) and 441 (magenta).

Nomenclature

b — number of balls.

h — maximum throw height.

n_{s} — total number of states.

n_{snt} — number of nontrivial states.

s, s_k — juggling states.

s_{g} — ground state.

$\mathcal{S}$ — set of juggling states.

t, t_k — transitions between states.

$\mathcal{T}$ — set of transitions between juggling states.

References

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